Mathematical Interpretation of the "Science of Logic" of G.W.F. Hegel

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Abstract:

We maintain that dialectical logic does not need mathematical interpretation, because it is already formalized in mathematics. In this article, we will show that the meanings of basic mathematical abstractions can be explained by the concepts of Hegel's logic, that mathematics itself is an applied dialectical logic, and dialectical logic is an ontology of mathematics. The above thesis will be supported by a dialectical analysis of the Euler Identity, which is considered one of the basic abstract theoretical constructs of mathematics.

Keywords: dialectics, mathematics, ontology

Introduction

In the 20th century, Hegel's logic remained outside the development of modern cognitive science. We will not investigate here in depth the reasons for this. It may be due to the thriving tradition of analytical philosophy during the last century, which examined the dialectic way of thinking as speculative, "not scientific."[1] On the other hand, continental philosophy moved away from the philosophy of science: it developed more in the context of existentialism, phenomenology and hermeneutics. To many scholars it seems that the way Hegel's logic is exposited makes it almost impossible for its mathematical formalization. Hegel's attitude to mathematics also contributes to this, as noted by Alan Paterson.[2] Yet many authors believe that mathematical formalization of Hegelian logic

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[1] Bertrand Russel, *A History of Western Philosophy* (New York: Simon and Schuster, 1945), 730.

[2] Alan L.T. Paterson, "Does Hegel Have Anything to Say to Modern Mathematical Philosophy?", *Idealistic Studies* 32.2 (2002): 143-158.

is possible. Yvon Gauthier states, "It is important to notice that dialectical logic in the Hegelian sense is, in principle, amenable to a formal treatment."[3] Other authors such as Kredik and Shpenkov[4] make efforts to develop algebraic models of the dialectics of Hegel, but apparently, a formal working logic model based on "Science of Logic" does not yet exist. It seems that such attempts are poised for failure, because what M. Kosok describes as a central meaning in Hegel's dialectics: the movement of the negation itself, not the concepts (moments) of this movement.[5]

Thus, Hegel's logic has two important properties:

(a) It cannot be formalized as algebra under well-defined conditions;

(b) It has an inherent functionality of introspection, which contradicts the fundamental theorem of Gödel for the incompleteness of formal analysis.

How can one overcome these seemingly insurmountable difficulties in the mathematical interpretation of Hegelian logic? Yvon Gauthier states, "With Hegel's logic, we are not given at first an axiomatic skeleton, an uninterpreted or semi-interpreted language, but a fully interpreted one. We are faced with the interpretation, and we have to make our way to the abstract framework."[6] However, any interpretation also represents an abstract structure that transforms a specific content in reasonably organized abstract knowledge about reality. The question arises: what is the specific content that we shall interpret to achieve a mathematical form of dialectical logic?

Traditionally, attempts to formalize dialectical logic are symbolic interpretations of the dialectical concepts in "Science of Logic." It proceeds from the idea that these concepts are kinds of dialectical axioms

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3 Yvon Gauthier, "Hegel's logic from a logical point of view," in Cohen, R.S., & Wartofsky, M.W., eds., *Hegel and the sciences* (Spring, 1984), 303-310.

4 Leonid G. Kreidik and George P. Shpenkov, "Philosophy and the Language of Dialectics and the Algebra of Dialectical Judgements," at https://www.bu.edu/wcp/Papers/Logi/LogiShpe.htm.

5 M. Kosok, "The Dynamics of Hegelian Dialectics, and Non-Linearity in the Sciences," In Cohen, R.S. & Wartofsky, M.W., eds., *Hegel and the sciences*, 311 -349.

6 Yvon Gauthier, "Hegel's logic from a logical point of view," 303-310.

that can be arranged and interpreted symbolically within some algorithms to achieve logical inference. However, it must be kept in mind that “Science of Logic” is the concreteness of Hegel’s idea how the dialectical logic should look. This is the Hegelian logic, the way in which Hegel thought the dialectics. The result of its symbolic interpretation can only be an abstract symbolic description of Hegel’s interpretation of dialectical logic. This will not be a mathematical interpretation of the dialectical logic itself. In this sense, it will not hold the validity of a universal, formally described knowledge that can be used as a tool for analysis in science. What is the particular form of the dialectical logic that would be a subject to productive interpretation?

The dialecticians, whatever their place within historical and cultural tradition, have always indicated that dialectics is a universal law of everything that exists. If this is true, we can turn our attention to any specific content and through its analysis deduce all dialectical concepts. In fact, this is the meaning of Marx’s reversal of Hegelian logic. Instead of displaying the dialectical concepts of Hegel’s abstract "pure being," Marx extracted them out of the concreteness of use-value in the economy. It can be said that “Capital” itself is something like a formal interpretation of dialectical logic through the terms of the economy. Such would also be the interpretation of the dialectical logic by the terms of mathematics, which interpretation should reveal isomorphic structural meanings between the two. We could take any particular mathematical reality and interpret it through the contents of dialectical concepts. The product of this interpretation will be the discovery of their dialectical analog. At the same time, these mathematical entities would represent a symbolic diagram or algorithm for dialectic inference. This product would also have a meaning as an ontology of mathematics. What would be a particular mathematical reality that can undergo such dialectical interpretation?

Euler's Identity

Euler's Identity occupies a special place in mathematics.

$$e^{i\pi} + 1 = 0$$

It is believed that it is a summary of all mathematics, as it presents the basic constants of mathematics—the number $\boldsymbol{\pi}$, the base of the natural logarithm **e**, the imaginary unit ***i***, *the one* and *the zero* as basic concepts in number theory, and all mathematical operations—addition, multiplication and exponentiation applied in the Identity only once. Thus, Euler's Identity brings together the reality of the whole mathematical continuum and is a good object for the application of dialectical analysis. This analysis will be in the form of a direct comparison of the mathematical definitions of the terms of Euler's Identity with the content of dialectical notions in Hegel's logic.

The Result of Euler's Identity: What is the Zero?

In mathematics, *the zero* is regarded as a special single set: the empty set. In many ways, it is unique. It is possible to have infinite sets with one or more elements, but only one is the empty set, which does not contain any element. On the other hand, the mere expression empty set is a contradiction—It presumes elements and, at the same time, it lacks any of them. On the other hand, in terms of ontology *the zero* is a specific object, some *being*, named *zero*. However, being an empty being, it is devoid of content, it is *nothing*. In other words, in terms of ontology, the zero is also a contradictory notion—it is both *being and nothing*. That, however, is the definition of Hegel's pure being, with which he began his "Science of Logic": "Being, pure being, without any further determination... the indeterminate immediate, is in fact nothing, and neither more nor less than nothing."[7]

The Nature of the one

We see that in mathematics and the dialectical logic of Hegel *the zero* has essential common definitions that match, and which are contradictory in their content. Hegel's dialectical logic uses this contradiction as a basic internal principle for its development and the emergence of the dialectical concepts. In fact, such a thing happens in mathematics, too. Using the same dialectic principle of negation, Georg Kantor constructs his Set Theory, displaying *the one*, the first non-empty set, as a negation of *the zero*: the singleton set is a set with one element and this

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[7] G.W.F. Hegel, *Science of Logic* (Cambridge: Cambridge University Press, 2010), 59.

element is the empty set. From the standpoint of dialectical logic, that is constructing Hegel's concept of *determinate being*. A being that is no longer nothing, pure uncertainty, but existence, which is definitely being different from nothingness: "Existence proceeds from becoming. It is the simple oneness of being and nothing. On account of this simplicity, it has the form of an immediate. Its mediation, the becoming, lies behind it; it has sublated itself, and existence therefore appears as a first from which the forward move is made. It is at first in the one-sided determination of being; the other determination which it contains, nothing, will likewise come up in it, in contrast to the first."[8]

Hegel argues that in *being* is also hidden its *negation*—the nothing, because without it *being* will also become *nothing*. However, in the same way, the notion of the singleton set in mathematics cannot be held without *the zero* as its main element: without *the zero*, this will also become an empty set, zero, nothing.

This dialectical logic, that is at the core of Hegel's "Science of Logic," and the foundations of Cantor's Theory of Sets, has a different fate in Hegel's philosophy and in mathematics. Once the singleton set in mathematics has been constructed, it becomes the object of standard mathematical operations, through which the entire mathematical continuum unfolds without the need explicitly to follow the dialectical principle of negation: the existence of contradiction as an internal engine of rational thought. Every contradictory object is explained as non-contradictory in another mathematical space of a higher order. Thus, Cantor's Theory of Sets develops as long as it does not reach another contradictory boundary of mathematics—the infinite set of all infinite sets. Since it is not possible anymore to construct a larger mathematical continuum than infinity, the contradiction that lays in the foundation of the Set Theory is again explicitly visible in the form of the famous antinomies of Cantor and Russell. It turns out that the beginning of mathematics—*the zero*—and the end—the infinite set of all the infinite sets in Set Theory—are contradictory objects. Let us see how these contradictions are dealt with by Hegel.

According to Hegel, *the nothing* is opposite, and at the same time identical to *being* in *pure being*. From the point of view of formal logic,

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8 Ibid., 83.

this is an absurd statement. Let us see, however, whether this is so for mathematics. If *the one* represents *a negation* of the empty set, then *the nothing* should also have such a *negation* that turns into being—*determined being*—as *a negation of the one*. Thus, it turns out that *the zero* as a contradictory mathematical object contains two opposing forms of being—positive and negative—one and minus one. The very expression that *the zero* contains something is itself a contradiction, but in mathematics, it is considered quite a rational contradiction: the unification of +1 and -1 is 0. Therefore, it turns out that in terms of dialectical logic *the zero* can also have another definition: the set of all infinite sets and their negations. Interestingly, in this way *the zero* coincides with the infinite set of all infinite sets: it is not only an empty set, but it contains all opposites. The zero is an infinite set, which contains infinite sets, but by definition contains itself because all sets are its own negations.[9]

The Contradiction—The Driving Force in Dialectics Recorded Symbolically by the Imaginary Unit i.

It is important, however, to emphasize that the extraction of the singleton set in mathematics and existence in dialectics rests on contradiction. The essence of this category, of central importance for dialectics, was first given to us in "Science of Logic" in the category *becoming*.

According to Hegel "Pure being and pure nothing are therefore the same. The truth is neither being nor nothing, but rather that being has passed over into nothing and nothing into being—'has passed over,' not passes over. But the truth is just as much that they are not without distinction; it is rather that they are not the same, that they are absolutely distinct yet equally unseparated and inseparable, and that each immediately vanishes in its opposite. Their truth is therefore this movement of the immediate vanishing of the one into the other: becoming, a movement in which the two are distinguished, but by a distinction which has just as immediately dissolved itself."[10]

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[9] This dialectical concept of zero was developed in the work of Bulgarian philosopher Ivan Punchev, *Introduction to dialectical logic*, without translation available in English.

[10] Hegel, *Science of Logic*, 83.

This is the content of the dialectical concept of *becoming*, described by Hegel. In mathematics, this process of transition from being to its opposite is described symbolically by the *imaginary unit* ***i***. *The becoming* itself is also a kind of *being*—it is something definite, something "being." However, according to Hegel, it is the pure expression of the contradictory nature of *the being* that constantly turns into its opposite, in *nothing*, and vice versa. Similarly, *the imaginary unit* is at the same time +1 and -1: that is why it is called *imaginary*, a non-existent, some strange thing in mathematics. Nevertheless, it does work, and it is in the very foundations of mathematics: without it all modern mathematics and physics would be impossible. Represented on the numerical axis, *the imaginary unit* also demonstrates that it is the opposite of the real numbers, of the individual beings—the axis of the imaginary numbers is perpendicular to the axis of the real numbers. Therefore, the construction of the continuum of complex numbers actually just illustrates the transition between *being* and *nothing* in the dialectical logic of Hegel, the "becoming."

Examined here from the perspective of the dialectic, nature of the zero reveals the essential meaning of the outcome of the Eulerian Identity:

$$-1 + 1 = 0$$

or nothing and being united give content to *the zero*.

What is the Nature of Nothingness?

How does *the negative form of being* (-1) occur?

Taken together, elements of complex numbers summarize the content of Hegel's evolving abstract pure being:

- *The* zero, center of the plane of complex numbers as a union of *being* and *non-being*;
- +1 and -1 as symbols of *being* and *non-being;*
- *The imaginary unit* as a mathematical expression of *becoming*, as being directly contradictory, as a transition between *being* and *non-being*.

In Euler's identity -1 is the product of the limit of the natural logarithm of the product of the number π and *the imaginary unit* ***i***:

$$e^{i\pi} = -1$$

Let us see the dialectical meaning of this mathematical expression.

What is the meaning of number π?

As it is known from mathematics, the number $\boldsymbol{\pi}$ is a mathematical constant expressing the ratio between the circumference of a circle and its diameter. This simple explanation is sufficient for the foundations of mathematics. Nevertheless, what does this explanation tell us? Why does this constant relationship between the length of a circle and its diameter exist at all? In other words, what is the ontological meaning of the number π?

If we look at the plane of complex numbers and its elements we will actually see that they construct a circle where zero is its center, +1 and -1 form its diameter, and complex numbers between them form the perimeter of the circle. Translated into the language of Hegel's dialectics, this means the following:

- *Being* is the distance from any point on the axis of real numbers to the circle's center, to the pure being, *the zero*;
- As such distance, it is already a being, but a *determinate being*, a certain *quality*;
- Every *quality* has its opposite being—its negation;
- The transition between this *determinate being* and its non-being *becomes* as *the quality* refers to itself as something else;
- This relation of *the quality* to itself as something else expresses the degree of *becoming* given by the imaginary unit $\boldsymbol{i}$ as a transition between *being* and *non-being*.

According to Hegel, *the quantity*, unlike *the quality*, finds its definition in itself, and not in relation to other qualities: "Plurality is posited in continuity as it implicitly is in itself; the many are each what the others are, each is like the other, and the plurality is, consequently, simple and undifferentiated equality. Continuity is this moment of self-equality of the outsideness-of-one-another, the self-continuation of the different ones into the ones from which they are distinguished."[11]

Quality X (number 1 for example) is different with respect to some other quality Y (number 2 for example), or Z (any other number on the real axis), etc. *The quantity* is just the number of referrals of one *quality*

[11] Ibid., 154.

to itself. Presented by geometry, *the quality* is any section along the axis of real numbers from zero to the corresponding number. *The quantity* is the perimeter of the circle described by *the quality* as the circle radius. However, Hegel also makes an additional step that reveals the meaning of the number π: both *the quality*, the distance between +1 and-1, and *the quantity*, the perimeter of the circle, are just two different definitions of *the determined being*, and just because they are different, they relate to each other. The ratio of quantity to quality in Hegel's logic is called *measure of being*: "Abstractly expressed, quality and quantity are in measure united."[12]

At the same time, every *being* is a *singularity*. We can have infinitely many singularities, but their essence remains always constant—this is their common *measure*. Therefore, in terms of dialectics, the number $\boldsymbol{\pi}$ is the quantification of any singularity—this is the *measure of the singularity* as ontological object.

After determining *the measure of singularity*, the dialectical logic defines the measure of its opposite—*the measure of multiplicity*. Any quality finds its definition in relation to another quality. Thus, each circle as a symbol of a single *determinate being* becomes defined in relation with any other circle. However, the question arises, what is that being between them, between the two circles? Here again we find a striking correspondence between mathematics and dialectics. What is between two qualities or numbers is *the boundary* between them, which is a mutually *shared being*. In mathematics it is expressed through the base of *the natural logarithm*

$$e = \left(1 - \frac{1}{N}\right)^N$$

In terms of dialectics, this math expression reveals the border of *reality* of any *singularity*: it is determined by the number of other qualities to which it relates. The more are these qualities (numbers), the more N grows, and the more reality of a single being tends to infinite determination based on the natural logarithm. Speaking philosophical language this is the transition between *abstract* and *concrete*: the small N (number

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[12] Ibid., 282.

of related qualities) means *reality* that is more abstract. And vice versa—the more related qualities we have, the more specific is the reality. Here we again find the relation of quantity to quality as it is in Hegel's logic: the base of the natural logarithm is *the measure of reality*.

Having clarified ontological definitions of the terms in Euler's Identity, let's see how they form a common semantic integrity.

The base of the natural logarithm raised to the power of product of **π** with *i* according to mathematics gives -1 as

$$\cos\pi = -1 \text{ and}$$

$$\text{but } e^{i\pi} = \cos\pi + \sin\pi$$

Speaking the language of dialectics it can be translated as *the measure of each singularity, powered by the maximum degree of its contradiction, turns into its nothingness*.[13] In other words, each *singularity* as a *controversial reality* always becomes unreality, nothingness. Placed as *uncontroversial reality* it resides in its positive or negative forms as a *being* or *non-being*. The unification of the two opposite forms of every single available being gives the definition of being at all—*the existence*, which finds its definition in continuous transition between the opposite states, in the motion that never arises or disappears, only manifests in its two opposing forms—genesis and disappearance. Neither of them can be determined without the other one.

Similarly, in mathematics the positive integers find their conclusion in the negative integers; integral calculus in differential; addition in subtraction; multiplication in division, etc. It turns out that all mathematics can be described as groups of symmetries and asymmetries that are isomorphic as abstract descriptions of the reality of the dialectical concepts of being and non-being, singularity and plurality, concrete and abstract, but as we will see in another study, they are united by two other fundamental common for mathematics and dialectics concepts—*finite*

[13] In fact, this is the meaning of the law of non-contradiction truth of Aristotle: if something is contradictory it does not exist, it is imaginary, until it becomes one of its polar definitions. It can be seen that logical laws for consistency of truth are special case of the contradiction as such, reflected in the mathematics of the imaginary unit *i*.

and *infinite*.

There is another, even deeper isomorphism between mathematics and dialectics. The development of dialectical concepts in the "Science of Logic" of Hegel is a going through the application of the basic principle of the dialectical inference. In the dialectical inference a concept (thesis) is represented by its negation (antithesis) in the form of mutually excluding polar differences, the negation of which (synthesis) is the transformation of the differences into one another and merging them again in an actual infinity.

The same scheme we find in the theory of groups in mathematics. According to Hermann Weyl, the term for a group of transformations is "any system $\mathfrak{G}$ of transformations of a given point-field, which is closed in the sense of the following conditions:

1. It contains the identity;
2. If S belongs to $\mathfrak{G}$, then its inverse S^{-1} does also;
3. The resultant TS of any two transformations S, T of $\mathfrak{G}$ is also transformation of $\mathfrak{G}$."[14]

Indicative is one of the examples that Weyl gives for a group of conversion: "A kinematic example of a group is offered by the motions of a space-filling substance, in particular those of a rigid body. The positions or numbers of the preceding example are here presented by the material points and the point-field is the space itself. The one-to-one correspondence p-p' connects the initial with the final state: that material point, which originally covered the spatial point p is taken to the point p' by the motion. Congruent correspondences of space onto itself will also be briefly referred to as 'motions' in the geometric sense."[15]

Interestingly, with this example, Weyl gives a definition of the movement, which is based on the difference between space being (point-field in the language of mathematics, the infinite in the language of dialectic, the continuum both in the dialectics and mathematics) and being of a body (material point in the language of mathematics, the finite in the language of dialectics, the discrete in both languages of dialectics and mathematics). Initially they match each other, but then by trans-

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[14] Hermann Weyl, *The Theory of Groups and Quantum Mechanics* (Courier Corporation, 1950), 112.

[15] Ibid., 111.

forming into themselves they split only to match again at the end by transformation of the transformation. Similarly, Hegel says, "This identity with itself, the negation of negation, is affirmative being, is thus the other of the finite which is supposed to have the first negation for its determinateness; this other is the infinite."[16]

The material point in the example of Weyl is the finite shape of the space, the volume of the rigid body, which is the negation of the infinite continuum of the space, the product of its transformation into something finite and discrete. However, the negation of this finiteness by reverse transformation leads to recovery of the identity of the space with itself in the new spatial position of the rigid body (material point, finite, rigid body). It could be said that the mathematical groups of transformations are a symbolic theory of the dialectical logical rule of negation of the negation.

Discussion

The question that arises is what is the heuristic meaning of thus established identity between mathematics and dialectics?

So far, mathematical logic always has been developed within the paradigm of consistency of truth, which dates back to Aristotle. It could be said that this paradigm has created one of the deepest crises in mathematics reflected in the antinomies of Cantor and Russell. As a logical consequence, it was further developed in the cognitive pessimism of Gödel's theorems of incompleteness of formal-logical knowledge. In practical terms, these seemingly abstract theoretical problems are transformed into the current failure to model the human mind. Ultimately, the human mind applies the mathematical abstractions into concrete computational operations, but we still do not have an ontological model of the human mind which creates and comprehends all scientific theories. We can free mathematics from its limitations imposed by the paradigm of consistency of truth (as we have seen, it is actually a special case of the dialectical truth), that could result in a widening of its fields of application in areas that now are still far away from its range: mathematical modeling of society (not statistical descriptions of social phenomena) and human intelligence.

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[16] Hegel, *Science of Logic*, 108.

There is also another very important aspect of the proposed mathematization of dialectics and dialectization of mathematics. The above dialectical interpretation of Euler's Identity revealed isomorphic key concepts in Hegel’s logic and the elements of the plane of complex numbers in mathematics. As noted above, the next step requires comparative analysis of the concepts of *finite* and *infinite*. In mathematics, the product of the inclusion of a point of infinity in the plane of complex numbers is *the Riemannian Sphere*: a mathematical construct of great importance to modern physics. Dialectic interpretation of *the Riemann Sphere*, which we will cover in another study, could serve as a means of developing new theories about the structure of the universe, both at cosmological and quantum level.