I. ANCIENT GREEK PHILOSOPHY

The Influence of Zeno's Axiom on Plato's Doctrine of the Fundamental Geometrical Figures

Goran Ružić (University of Niš, Serbia),
Biljana Radovanović (University of Niš, Serbia)

Abstract

In order to defend Parmenides' doctrine of the One in terms of a homogeneous, continuous, and indivisible being, Zeno argued that, if a multitude were possible, it would have to consist of a plurality of units (points), which is impossible because the units (points) of the multitude have no magnitude. That to which the units would be added could not increase just as that from which the units would be subtracted could not decrease. In *Metaphysics*, Aristotle named this proposition "Zeno's Axiom." This being said, Plato's ontology is based on two principles: the One and the Multitude (the Indeterminate Dyad). The introduction of multitude into the constituency of the whole of existence implies a direct negation of Zeno's Axiom, which, in turn, shows the paradoxical nature of such introduction. Our paper is concerned with the influence of Zeno's Axiom on Plato's philosophy, especially on his theory of the fundamental geometrical figures, i.e., points, lines, planes, and bodies.

Keywords: Zeno's axiom • point • line • plane • body • geometry • Plato

1. Introduction: Zeno's Axiom

In order to defend Parmenides' doctrine of the One in terms of a homogeneous, continuous, and indivisible being, Zeno argues that, if a multitude were possible, it would have to consist of a plurality of units (points), which is impossible because the units (points) of the multitude have no magnitude. In order to defend Parmenides' doctrine, Zeno criticizes the position of those who attack him. First of all, he attacks the Py-

thagorean conception of the existence of a multitude of units. He demonstrates the deficiencies of the Pythagorean doctrine by means of a *reductio ad absurdum*. In *Metaphysics*, Aristotle uses Zeno's Axiom to argue that there is no such thing as an indivisible unit of multitude.[1] Unlike Zeno, Aristotle uses the term στιγμή (*stigmē*) to denote the point as an indivisible unit. This means that the unit does not exist unless it is indivisible.[2] Zeno is not primarily interested in defending the assumption that that which lacks extension does not exist, but rather in finding "an adequate candidate" for the constituent of the multitude. As such, the point is nothing since it has no magnitude, thickness, or mass. If such unit lacking extension were added to something, it would not increase it. Nor would any subtraction decrease it. In other words, according to the mathematical interpretation of Zeno's Axiom, the addition of a point to a point cannot result in a line, just as the addition of a line to a line (in concatenation) cannot result in thickness. The same is true of the addition of a plane to a plane or the subtraction of a plane from another.

2. Definition of the Point

In the first definition of Book 1 of *The Elements*, Euclid holds that: "The point is that which has no parts."[3] Obviously, Euclid considers a geometrical point to be an entity that lacks breadth and length. In other words, it lacks dimensions and is therefore to be considered zero-dimensional. A well-known German classical philologist, Hermann Fränkel, insisted on the difference between nothing and units lacking magnitude, i.e., points.[4] He thinks that the Eleatics concluded that, if something is annulled, it does not exist. If the point is annulled as a zero-dimensional entity, does it mean that it does not exist? Mathematicians would have no great difficulty answering this question, since for them the concept of point is one of

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1 Aristotle, *Metaphysics*, in *The Complete Works of Aristotle*, trans. W. D. Ross, Princeton New Jersey: Princeton University Press, 1991, 1001b10–26.

2 Ibid., 1001b10–26.

3 Euclid, *The Elements*, in *The Thirteen Books of Euclid's Elements*, translated with introduction and commentary by T. L. Heath, vol. I, Introduction and Books I, II, Cambridge: Cambridge University Press, 1968, p. 153.

4 Hermann Fränkel, "Zeno of Elea's Attacks on Plurality," *American Journal of Philology*, vol. 63, n. 1, 1942, pp. 1–25; vol. 63, n. 2, 1942, pp. 193–206.

the fundamental concepts of the axiomatic system in Euclidian geometry. For a contemporary mathematician, the word "point" is meaningful only in reference to the system to which points belong. The difficulty rather arises with ontologists, who have to determine the ontological status of the point: in which manner and in which sense does the geometrical point exist? Or, following Plato, in which realm can a geometrical point be found? Can it be found in the sensible (ορατὸυ) or in the intelligible (νοετόν) realm? Can we perceive a geometrical point in physical space or can we only conceive it mentally?

Let us repeat the simplest version of Zeno's Axiom in light of these questions: *An indivisible magnitude is nothing, it does not exist*. This is because any magnitude is divisible *ad infinitum*. Thus, there are no indivisible magnitudes. Historically speaking, Zeno's Axiom was not directed against the thesis that there exist points that lack magnitude, i.e., points that have no parts, indivisible points. Instead, it was probably directed against the Pythagorean doctrine of the geometrical unit, i.e., the point with an indivisible magnitude.[5] It is impossible for a spatial magnitude to consist of indivisible spatial magnitudes. Using propositional logic, the previous claim can be demonstrated as follows:

1. If the point as a geometrical unit had magnitude, then it would be divisible ($P \rightarrow Q$)
2. The point is not divisible ($\neg Q$)
3. Therefore, according to *modus tollens*, the point has no magnitude ($\neg P$)

Let us turn to Aristotle's account of the senses of the word "one": "we say that the continuous is one or that the indivisible is one, or things are said to be one, when the account of their essence is one and the same, as liquor and drink. If their One is one in the sense of continuous, it is many; for the continuous is divisible *ad infinitum*."[6] By means of

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[5] In a two-dimensional Euclidean space, the point is represented in an ordered pair (x, y) where x and y stand for numerical values, i.e., for a position at an abscissa and ordinate.

[6] Aristotle, *Physics*, in *The Complete Works of Aristotle*, trans. R. P. Hardie and R. K. Gay, Princeton, New Jersey: Princeton University Press, 1991, 185b8–12.

this semantic distinction, Aristotle adduces that being must have a magnitude, and a continuous and unlimited one at that. In the Pythagorean conception of the fundamental geometrical figures, a point is considered a monad occupying a location and the number one is seen as a point (monad) without a location. Aristotle was not prone to accept such a conception of the point and, therefore, criticizes it: "Moreover, there can always be something between points (for all lines are intermediate between points), whereas it is not necessary that there should be anything between units; for there is nothing between the numbers one and two."[7] The phenomenon of the point is that which distinguishes geometrical space from the physical one. The difference between a geometrical and a physical point is that the former has no extension, whereas the latter does. Also, the geometrical line has no width (thickness) but serves as a surface boundary. In physical space, a position is found in the real part of that space and is as such dimensionally determined as space. Unlike points in geometrical space, those in physical space (physical points) have a magnitude. In geometrical space, there is no juxtaposition (parallel positioning) in which a line could arise from a series of points and in which a plane could arise from lines or a body from planes. A zero-dimensional entity such as the geometrical point cannot produce a unidimensional entity such as the line. The basic difference between the point and the (Pythagorean) monad is its location. For the Pythagoreans, due to the theory of juxtaposition, location is related to magnitude. In this respect, to say that "a point is a monad with a location is the same as to say that "a point is an extensive monad" and, *vice versa*, "a monad is a point without location or "a monad is a non-extensive point."[8] It is only in this sense that the Pythagoreans could be said to have understood the point in terms of an atom of geometrical space.

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[7] Ibid., 227a31–34.

[8] Transferring the testimony of the doxograph Aetius, Arsenijević says that Ecphantus from Syracuse claimed that "the Pythagorean point is corporal" and that, apart from empty space, only indivisible bodies exist. Miloš Arsenijević, *Prostor, vreme, Zenon*, Beograd-Zagreb: Biblioteka Teka, 1986, p. 110.

3. The Theory of Indivisible Lines

Ancient commentators (doxographers and Athenian chronologists) frequently report the attitudes of the proponents of the theory of "indivisible lines" (i.e., atomic lines). However, it is not completely clear whether these reports refer to Plato or Xenocrates (the Academy disciple). Nevertheless, besides Aristotle, who is the main source for the "indivisible lines" problem (geometrical atomism), in this paper we will also use the Peripatetic or pseudo-Aristotelian text entitled "On Indivisible Lines" (Περὶ ἀτόμων γραμμῶν, *De lineis insecabilibus*).[9] Our primary concern with this text is not its authenticity, but the issue related to the object of the Aristotelian criticism. At any rate, we must bear in mind that it is one thing to provide an account of indivisible lines and a totally different thing to accept their existence. This might seem paradoxical, but it is our task, for the purpose of this paper, to reconstruct a part of Plato's doctrine of indivisible lines that cannot be found in his work, but only in indirect testimonials of the doxographers from Antiquity, alongside Aristotle. Of course, what we are referring to is known as Plato's unwritten doctrine taught through academic lectures that bear the common title "On Goodness."[10] We will also be referring to parts of *Timaeus*.

If we can show that Plato believed that it was the indivisible lines and not the points that constitute the basic elements and the beginning of the straight line, that may prove that he advocated for geometrical atomism of a kind different from the one described in *Timaeus*, which deals with indivisible planes. Aristotle mentions that Plato frequently spoke of indivisible lines in his lectures, in which he considered the points in terms of geometrical function, and that he regarded the indivisible lines to be the beginning of the line.[11] In the section we just referred to, Aristotle writes about the beginning of (things in) existence and the problem of the lower genera being contained in the higher. His aim to account for the originary principle in the construction of nature, just as it is the case with the ontology of Plato's unwritten doctrine about the One and the

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[9] Aristotle, *Metaphysics*, 992a10–23; *Physics*, 231a20–25. These are just some of the places in which he discussed ἀτόμων γραμμῶν.

[10] H. J. Krämer, *Plato and the Foundations of Metaphysics*, trans. J. R. Catan, New York: State University of New York, 1990, pp. 96–97.

[11] Aristotle, *Metaphysics*, 992a10–23.

Indeterminate Dyad. For Plato, a line is derived from "the long and the short," a plane from "the broad and the narrow," and a body from "the deep and the shallow." This means that a line occupies one dimension, a plane occupies two, and a body occupies three. Since the line, plane, and body belong to different genera, the question arises as to the manner in which a plane contains a line, and a body both a plane and a line.

Some believe that Plato formulated a continuum theory according to which continuous (geometrical) objects are those having common limits, which means that the limit of one (object) is the beginning of another. According to A. T. Nicol's interpretation of Plato, the end of a body is a plane, the end of a plane is a line, and the end of a line is a point.[12] This interpretation contradicts the hypothesis of the fictional point. Contrary to this, it should be stated that the point is the beginning of a line, i.e., that Plato recognized points in terms of the first indivisible lines. However, in this way we cannot avoid answering the question whether the point, being the beginning of a line, contains any magnitude. Conversely, if a body contains a plane, a plane contains a line and a line contains a point, we have to acknowledge the thickness of the plane, the width of the line, as well as the length of the point, which is impossible. Hence, as Miloš Arsenijević remarks, "the points would be corporal entities and still completely indivisible, divisible lines would include one and only one type of divisibility, whereas the divisible planes would be divisible in one and indivisible in two ways. Bodies that do not comprise planes would not be divisible in any respect."[13]

Plato distinguishes the point understood as a geometrical fiction from the point understood as the beginning of a line, i.e., points as indivisible lines. The hypothesis of the existence of indivisible lines amounts to the assumption that the entities of a certain dimension can be derived from the entities of the same dimension: a line from "shorter" lines, a surface from planes, a space from volumes. The starting point is the assumption that indivisible lines must exist since, provided that an infinite division is allowed, that which is small could become infinitely big. Besides, there is one indivisible Idea of a line, since, if it were divisible, its parts would have a logico-ontological prevalence over its totality.

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[12] A. T. Nicol, "Indivisible Lines," *The Classical Quarterly*, 1936, pp. 123–126.
[13] Miloš Arsenijević, *Prostor, vreme, Zenon*, p. 111.

Finally, Zeno's kinematic paradoxes (proofs against motion) show that infinity cannot be covered step by step, regardless of the (racer's) speed. Therefore, indivisible lines must exist. Otherwise, infinity could be covered step by step in a limited time span, which is impossible. The crucial evidence is mathematical: indivisible lines are derived following the commensurability hypothesis. However, a strong counterargument to this mathematical proof is the phenomenon of incommensurability through which the indivisible lines theory is refuted. Geometrical atomism could not be justified through the use of the physical indivisibility of atoms even if the validity of physical atomism were accepted.

A. T. Nicol is right when she claims that Plato's *Timaeus* is not relevant for the indivisible lines hypothesis, since the dialogue deals with the physical world in which there is no need to go beyond indivisible planes.[14] The author of the treatise *On Indivisible Lines* himself invokes Zeno and his argument against multitude having commensurability of all the lines.[15] Also, he directly expands upon Zeno's proofs against movement with the following argument: individual lines must exist because, otherwise, infinity could be covered step by step in finite time, which is impossible.[16] That which represents a novelty in comparison to Zeno's original proof is the introduction of the "speed of thought" as an additional factor the purpose of which is to enhance the argument, since not even the thought that is unsurpassed in speed can be used in counting an infinite set step by step.

The hypothesis of indivisible lines functions as a sort of "prevention" against potentially infinite geometrical division. In the last analysis, to say that the division of the straight line ends in the point is to say that, in fact, it does not end at all. The infinite division of a finite entity, e.g., a straight line, turns that finite entity into an infinite one, which is absurd. The argument is a *non sequitur*: the fact that infinity cannot be covered step by step does not entail the actual existence of indivisible lines. Aristotle contradicts the proponents of the indivisible lines theory in the following way: "It is surely absurd that, because you are unable to solve Zeno's argument, you should make yourselves slaves of your inability, and should commit yourselves to still greater errors, in the endeavour to

[14] Nicol, "Indivisible Lines," p. 125.
[15] Aristotle, *On Indivisible Lines*, 968b5.
[16] Ibid., 968a18–968b3.

support your incompetence."[17] Aristotle considered the question of the constitution of geometrical dimensions and of higher-order objects through those of the lower order (the possibility of constructing the straight line out of a point or an indivisible line) to be a matter of the relationship between continuous and discrete magnitudes.[18] A straight line is a continuum, but it does not consist of points which, by definition, have no magnitude. The points cannot be continuously tangential but only contiguously so (as physical points). However, in that case the series would be a discrete one, not a continuous one, which means that it would not comprise a line at all. Just as a set of punctual moments cannot create a temporal interval, a set of points cannot create a line. The proof of the impossibility of the existence of indivisible lines is based on the empirical fact of the continuity of time and space in their mutual interconnection. The continuum of any kind is that which is by itself infinitely divisible.[19]

The result of Aristotle's analysis is that the continuum of any kind cannot be constructed out of indivisible elements (i.e., indivisible lines). Nicol thinks that Plato's continuum is an ordered system of limits that enables and necessitates the transition of the entities within a genus. It is a continuum based on the principle according to which the end of one genus is the beginning of another.[20] Plato's ontologization of the totality of being is overarched by the principle of the One and the Indeterminate Dyad (multitude), then by the Idea of numbers and ideal dimension, and, ultimately, by the visible phenomena in the world of senses. The consequence of the acceptance of indivisible lines implies the reduction of geometrical entities to numbers. Hans Joachim Krämer points to Plato's unwritten doctrines in which mathematical numbers, geometrical, and stereometric objects are construed out of monads and minimal magnitudes (indivisible lines).[21] According to those doctrines, the monad constitutes the first principle of numbers, a divisible line is the principle accounting for all types of movement, and the moment is the principle of time.

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[17] Ibid., 969b4–5.
[18] Aristotle, *Physics*, 231a29–b6.
[19] Ibid., 231b15–16, 232b24–26.
[20] Nicol, "Indivisible Lines," p. 123.
[21] H. J. Krämer, *Plato and the Foundations of Metaphysics*, pp. 84–85. It is clear that in the field of geometry Plato favors the expression "indivisible lines" since the mathematical point was defined as a limit index without extension, whereas in physics the expression "punctual monad" was used.

4. Indivisible Lines Form a Continuum

Many difficulties could be solved by the elimination of Zeno's hypothesis of infinite division and the introduction of minimal indivisible magnitudes (atoms, indivisible lines, infinitesimals) for space and time. However, that is impossible. The key problem arises once we ask the following question: do the minimal, absolutely indivisible parts of space have a form? If they do, infinite division is allowed. If they do not, the minimal indivisible parts do not exist. As Arsenijević says, "in mathematical terms a continuous magnitude cannot have a part which is not itself a continuous magnitude, since therein new parts can be discovered which are continuously [geometrically — authors' addition] tangential. The contact of the elements cannot be realized through the insertion of geometrical points, because, as Zeno suggests, contact is impossible since geometrical points have no extension. If the division treated in Zeno's fragment is understood as halving (division in terms of dichotomy), then the end of the parts of things cannot be arrived at, that is, the plane in question cannot be reached. In Arsenijević's words, "[i]f the division is understood as a trichotomy, in which case the middle part decreases and the central point is to be reached beginning from the ends, then the center cannot be arrived at."[22] Zeno's argumentation against multitude aims at demonstrating that, if a multitude exists, its elements have to be so small that it has no magnitude and so big that it is infinite. The point cannot be reached by division, but only by abstraction from magnitudes, and division itself, once allowed, can never be finite.

5. Interpretations of Zeno's Axiom

Pure geometrical figures understood as the limits of bodies cannot have an independent existence, but physical substances as concrete unities of matter and form can be ontologically independent.[23] This means that, ontologically, physical bodies have priority over planes, lines, and points. Aristotle is explicit in his claim that bodies cannot consist of points, lines, and planes.[24] Aristotle understands the point as a limit

[22] Ibid., p. 64.
[23] Aristotle, *Metaphysics*, 1042a24–b9.
[24] Ibid., 1077a34–35.

(πέρας), not as an element of a set that could be reached through endless decomposition. Not only was Zeno's Axiom accepted within the Euclidean conception of points, but it was not refuted until the publication of Cantor's and Dedekind's work, i.e., until the second half of the nineteenth century.

Arsenijević treats the question of the autonomy of points in terms of the difference between the standard (i.e., Aristotelian-Euclidean) and the non-standard (e.g., Cantor, among the firsts) interpretations of points. Accordingly, if Zeno's Axiom presupposes the independence of the entities of lower dimensions from which the entities of higher dimensions are composed, then this independence will hold only for the standard interpretation of points in which a point cannot be a part of a particular straight line. The problem of indivisible entities cannot be solved by introducing infinitely small magnitudes (infinitesimals) since, from the perspective of the infinitesimal method itself, it is necessary for the infinitesimals to be divisible. The discovery of a non-standard point on a non-standard straight line is impossible in non-standard space because it is isotropic. What is meant by isotropic space is the standard Euclidean-Archimedean two-dimensional space in which points have no extension and no shape.[25] Isotropic space by definition does not support Euclidean points out of which, according to Zeno's Axiom, the line cannot be composed.

Following Euclid, we defined points as geometrical entities without extension. In the same way, lines have no breadth and planes have no thickness. Accordingly, the limits of the parts of space are considered only continuously (geometrically), but not contiguously (physically) tangential. There must always be a finite number of actualized limits,

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[25] The German mathematician Cantor thought that his theory of sets of points refuted Zeno's Axiom by introducing the possibility of the straight line in terms of a linear one-dimensional continuum, and subsequently for the whole infinite *n*-dimensional space to be composed out of non-extensive zero-dimensional points in terms of set units. Grünbaum defines a particular space as zero-dimensional if any of its possible limits comprises an empty set, which is exactly the case with Cantor's discontinuum consisting in 2*n* points, but isolated ones. Adolf Grünbaum, "A Consistent Conception of the Extended Linear Continuum as an Aggregate of Unextended Elements," *Philosophy of Science*, vol. 19, n. 4, 1952, p. 294.

whereas the number of potential limits is infinite. Since dealing with pure space, some infinitists have wiped away the differences between points, lines, and planes as actual *versus* only potential limits. As Arsenijević says, "[w]herever they [i.e., infinitists] spot the situation in which unlimitedly many points can be produced, they immediately begin to talk about the givenness of infinitely many points... consciously ignoring the fact that in some of these situations, according to their own hypotheses, there literally is not enough space for the simultaneous realization of infinitely many points."[26] The intervals are treated the same way alongside points. This is shown in the formula $(2n - 1)/2n$, $(2n + 1 - 1)/(2n + 1)$, which indicates that there can be infinitely many intervals. However, some mathematicians think that there are infinitely many of them, because their number cannot be fixed insofar as a subsequent addition of more intervals is always possible. To mathematicians, space appears as a set of all the points that could be actualized in a potential division. The problem is that real space does not consist of points, but of parts that actualize a particular possible distribution of points. Points, lines, and planes are limits, and that which makes them limits cannot be that which they consist of. Mathematical space as a set of points is a field (*Bereich*) of possible limits that cannot (all) be actualized either simultaneously or successively. In Aristotelian-Euclidean terms, the basic geometrical entities were viewed in such a way that they were ultimately understood as limits of the heterogeneous parts of physical bodies. Through the redefinition of the concept "point," mathematicians (Cantor, among the firsts) defined any point in terms of a convergent series of rational numbers. In this way, for mathematicians, the continuum is a set of all possible points. However, it does not follow from this that the line is a set of points that can be discovered on it. It only follows that this kind of definition of points makes it possible for the points to be a part of a continuum, i.e., a straight line.

6. The Relation Between Higher and Lower Dimensions

Zeno's Axiom is a paradox: a multitudinous body that has been "subjected" to division emerges as an infinitely small and infinitely big one. It is infinitely small because through division we reach points without

[26] Miloš Arsenijević, *Prostor, vreme, Zenon*, p. 326.

magnitude. It is infinitely big because it consists of infinitely many parts with magnitude. Zeno is brilliant! His axiom is not only a proof against multitude, but also a defense of Parmenides' monistic ontology according to the law of excluded middle (*tertium non datur*). Apart from the manner in which division may be interpreted, one cannot reach anything more elementary than that from which division has started, since any part of that which is continuous must be continuous itself. In other words, anything that is (allegedly) a part must be as elementary as that of which it is a part. If we were to claim otherwise, i.e., if we were to claim that the continuum is composed of something that is not continuous, we would have a problem, because a mathematical point cannot be an essential part of any entity that has a dimension or dimensions.

The consequence of the acceptance of Zeno's Axiom is that, if the continuum (Parmenides' being) is homogeneous, undivided, and indivisible, then it is not structured.[27] Aristotle would add that the continuum is actually undivided, but that it is potentially divisible, and into infinity at that. Potential divisibility *in infinitum* does not imply the possibility of actual infinite dividedness.[28] To be even more precise, the continuum does not possess infinitely many actual parts, but does have potential ones. Alongside Parmenides, Aristotle understands the continuum (συνεχές) as a concatenation of the potential parts of something that is homogeneous (unified), but not of that which is heterogeneous (diverse), such as, for example, Homer's uninterrupted succession of day and night. If we were to accept the claim that there exists a multitude, we would also have to accept mutually contradictory propositions, which contravenes the principle of non-contradiction.

To sum up, Zeno should not be ascribed the view that a point is nothing, but rather that it is nothing only if seen as an essential part of a body. The claim that a point is nothing as a multitude constituent (zero-dimensional) does not imply that it does not exist. A geometrical point is in no way divisible, but it does exist, at least as a limit.

By raising the question of a point as the beginning or the principle of the line, we transition to the discussion of the line. It is noteworthy that,

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[27] It is irrelevant at this point of our discussion whether the division in question is actual (real) or potential (imagined).

[28] Aristotle, *Physics*, 206a17–25.

when Plato and Aristotle use the term γραμμή (line), they do not mean by that what we today consider a straight line, but a line segment of any other limited line that has a beginning and an end. The beginning and the end of line segments are points. Because the concept of a limited line entails the concept of a point, and because a point can be defined as something indivisible without magnitude, Aristotle allows for the logical order to follow from a point to a body.[29] Logical priority does not imply an ontological one. In other words, it is not because something is logically prior that it is also prior in terms of substance. Accordingly, Aristotle concludes that physical bodies have ontological priority over mathematical objects because they are (allegedly) substances in a greater degree. If it is possible to talk about geometrical substances at all, these exist as the external forms and limits of physical bodies. But it cannot be claimed that within a particular, unified, continuous physical body there exist points, lines and planes. What can be claimed is the existence of possible points, lines, and planes with regards to possible divisions in which they could be actualized. The number of those possible parts is indeterminate, but not infinite in the categorematic (actual) sense.[30]

7. The Relation Between Point, Line, and Surface

In his *Elements*, Euclid treats the line in general first ("A line is a breadthless length"), and then the straight line ("A straight line is a line which lies evenly with the points on itself").[31] In Book 1 of *Elements*, definition 6, Euclid provides one more definition: "The extremities of a surface are lines."[32] The introduction of the concept of line in definition 2 is, in fact, the introduction of the first dimension since a point cannot be an extensive magnitude. Aristotle defines the line as a continuum of length, where length is, according to him, a magnitude having one direction, breadth having two, and volume having three.[33] Since, in the ancient paradigm, a line, which has length, constitutes the first dimension, it represents the principle of magnitude in general (mathematical, logi-

[29] Aristotle, *Metaphysics*, 1077b30–1078a4.
[30] Miloš Arsenijević, *Prostor, vreme, Zenon*, p. 168.
[31] Euclid, *The Elements*, (I) def. 2, 4, p. 153.
[32] Strictly speaking, that is not a "definition," but rather an "example."
[33] Aristotle, *Metaphysics*, 1020a9–1020a14.

cal, ontological).[34] The ancient idea of a dimension corresponds to the contemporary idea of vector space, which means that in a three-dimensional coordinate system any point can be determined according to its distance from X, Y, or Z axis. Another idea, which has been ascribed to Speusippus, is that a line can be "obtained" out of a point without juxtaposition through a "thought experiment" in which a point is set in motion. In that case, a line would represent an abstraction of the path that has been crossed. The definition of the line in terms of breadthless length cannot be found in Plato's writings. However, his *Parmenides* contains a definition of the straight line: "straight is that of which the middle stands in between both extremities."[35] If Aristotle's expression "those who consider" refers to Plato or somebody from the Academy, then the definition of a line as breadthless length refers to the Platonists. It is not the definition itself that bothers Aristotle, but the fact that they (hypothetically Platonists) believe in the existence of a numerically one Idea or ideal genus of the line.[36] In fact, he analyses the specific difference of the genus as follows: if anything can be truthfully either predicated (attributed) or negated, then every length either must have or cannot have a breadth. This would mean that there are lengths without breadths (lines) and lengths with breadths (surfaces). The concepts "without breadth" and "with breadth" are *differentiae specificae* of the genus "length." We define a species by providing its genus and its specific difference. Here, the genus encompasses both species of length as well as their differences. If the Platonic genus is numerically one, then it cannot fulfil this function. Length in itself either has or does not have breadth and, if there is such an Idea of length, the question arises as to how this genus can have breadth. If the Idea of the line as breadthless length exists, then the whole genus of lengths will imitate that Idea. This means that the predicate will necessarily and truthfully be attributed to the whole genus of length, which is contradictory to the fact that the genus comprises both the lengths with and without breadth.[37]

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[34] In the same way that numbers begin with "two."

[35] Plato, *Parmenides*, trans. Arnold Hermann, Las Vegas/Zurich/Athens: Parmenides Publishing, 2010, 137e.

[36] Aristotle, *Topica*, in *The Complete Works of Aristotle*, trans. W. A. Pickard, Cambridge/Princeton/New Jersey: Princeton University Press, 1991, 143b11–12.

[37] Ibid., 143b11–34.

The Ancient Greeks classified the line in different ways. In their doctrine of opposites, the Pythagoreans accepted the pair of opposites "straight and curved." Euclid mentions only the straight line and the circle, but not the so-called "mixed lines" (curves in space and curved lines in planes). Another ancient classification is the one involving simple and complex lines. Simple lines are further classified as "limited," i.e., "lines forming the figures" and those that "extend into infinity." Closed lines are the circle, ellipsis, and cissoid, whereas the straight line, parabola, hyperbola, conchoid, and helix are infinite lines. Complex lines are those which "form angles." Mixed lines would actually comprise fractured lines that can be derived from all types of angles known to Ancient Greek mathematics. All types of lines are derived from two types of movement, linear and circular, whereas mixed lines are derived from their combination. In *Parmenides*, Plato divided shapes based on these two types of movement. The types of shape in question are "circular" and "straight" as well as the combination of the two.[38] The term Plato uses for shapes is σχὴμα (*schema*), which serves to denote a two-dimensional geometrical figure. Just as numbers can be odd or even, lines can be straight or curved. Aristotle thought that the curved line could either be reduced to the circle or be derived from its combination with the straight line. Regardless of whether a curve "contains a bend" or not, it is unified since it is uninterrupted. However, the straight line is "more unified than the curved one," since it is simultaneously continuous and without a bend.[39]

In light of Plato's statements from *Timaeus*, it can be inferred that the straight line is the symbol of all that which is unchangeable, incorruptible, unshakable, inflexible. The circle, however, and the circular movement is a symbol of autoreflection, i.e., an expression of philosophical thinking reminiscent of the production of Plato's demiurge.[40] With regards to Plato's unwritten doctrines, the circle would correspond to the principle of the One, the straight line would correspond to the principle of the Indeterminate Dyad, and their combination would correspond to the lines originating in the combination of translation and rotation (helix, conchoid, cissoid). The belief that all types of mathematical

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[38] Plato, *Parmenides*, trans. Arnold Hermann, Las Vegas/Zurich/Athens: Parmenides Publishing, 2010, 137d14–e1, 145b5–6.

[39] Aristotle, *Metaphysics*, 1016a13–17.

[40] Plato, *Timaeus*, in *Timaeus and Critias*, trans. Robin Waterfield, Oxford: Oxford University Press, 2008, 40b.

objects of the first dimension are derived by abstraction from the circular and/or linear movement has persisted up to this day. The Platonists' prioritization of the circle in comparison to the straight line is close to contemporary geometrical conceptions in which the straight line is regarded as the circle of an infinite radius.

In a two-dimensional Euclidean space, the straight line can be defined both implicitly and explicitly. The implicit (Cartesian) definition is expressed in the following equation: $ax + by + c = 0$. The explicit definition is illustrated by the equation: (x), i.e., $y = kx + n$, where x and y denote the values of the abscissa and ordinate, k is a quotient of the direction-tangent of the angle that the straight line closes at the positive part of the abscissa, n is a positive segment of the ordinate. A geometrical straight line is an abstraction of the direction that is *a priori* given, but still it is not completely independent from sensory experience. It is not wrong to say that the direction originates by way of abstraction from the experience of that which is straight.[41] The straight line would actually be the shortest possible distance, the line segment linking the eye to the object of vision. In this respect, the curved line would include a greater distance than the straight line. Plato was right when he suggested the definition of the straight line in terms of the philosophical analysis of sight: "straight is that of which the middle stands in between both extremities."[42] This is also the only definition of the straight line that precedes Euclid's *Elements*.

Mathematically speaking, in the orthogonal projection of the straight line p onto the plane π, if p is parallel to the direction of the projection s, the image p in the plane will be the point. If we imagined being able to see through the line segment from the perspective of the vertex, what could be seen would be the point. Plato's definition of the line is not a definition of the straight line. The contemporary concept of the geometrical straight line starts from the assumption that actual infinity (categorematic infinity) has been accepted, whereas the Ancient Greeks accepted only potential infinity (syncategorematic infinity). Within the Ancient Greek paradigm, a line is always finite, regardless of whether it is straight or curved. The line is actually limited and potentially infinite

[41] "Let the following be postulated: Postulate 1. To draw a straight line from any point to any point; Postulate 2. To produce a finite straight line continuously in a straight line." Euclid, *The Elements*, Postulates I.1 and I.2., 154.

[42] Plato, *Parmenides*, 137e5.

in terms of its capacity for further extension.[43]

Following Archimedes' assumption that the straight line (line segment) is the shortest possible distance from one point to another, Euclid provides the definition according to which the interval between any two points is the line's length which is determined by these two points. For the two points on any curved line, the length of the line between those two points will be greater than their distance. The only case when this does not hold is the straight line. This is exactly what Euclid means by the statement "A straight line is a line which lies evenly with the points on itself" (*Elements*, Def. I.4).

8. Intelligible Shapes and Geometrical Shapes

For Plato, mathematical entities are intelligible beings and, as such, they belong to the realm of *dianoia*, i.e., of what is known by means of rational knowledge. They are halfway between sensory phenomena that can be perceived and Ideas that can be reflected by the mind (such as the Good). Being imperceptible, mathematical entities are similar to intelligible Forms. However, being multitudinous, they bear a similarity to concrete things. Aristotle's references to Plato's teaching at the Academy reveal that the "late" Plato also introduced, besides mathematical numbers and magnitudes, something that could be called ideal numbers and magnitudes. Leaving aside the status of the order of ideal magnitudes, it is certain that we are dealing here with unidimensional, two-dimensional, and three-dimensional magnitudes, and not with geometrical magnitudes *per se*.

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[43] Euclid's definition of the straight line (Def. I.4) is essentially based on Plato's. Archimedes similarly defined curved and fractured lines based on the straight line. He thinks that any limited line whose beginning and end overlap with the beginning and end of the straight line completely lies either on the same side or on both sides of the straight line. Thomas Heath, *A History of Greek Mathematics*. *Volume 1 From Thales to Euclid*, Oxford: Clarendon Press, 1921, pp. 2–3.

9. Plato's Doctrine of Ontological Principles: The One and the Indeterminate Dyad

Aristotle reports that the Platonists explained the generation of lines on the basis of the dyad, that of planes on the basis of the triad, and that of volumes on the basis of the tetrad.[44] The three types of principles of the Indeterminate Dyad, namely "the long and the short," "the broad and the narrow," and "the deep and the shallow," are in fact three different aspects of one and the same principle, i.e., the principle of "the great and the small." At first glance, Aristotle's criticism of the principle of the Indeterminate Dyad seems successful. At this point, the principle is only relevant if it refers to the ontological background in the production of geometrical objects. The founder of the Lyceum starts from the assumption that "the great and the small," "the straight and the curved" are not principles, but properties. Accordingly, he analyses the consequences of the two different, but still mutually related judgements of the Platonists: if all these dimensions were generated by the principle of the Indeterminate Dyad, as well as if they were not, what implications would this have on the dimensions themselves and their mutual relation within the body? If the great and the small, the long and the short, the broad and the narrow are just different properties of one and the same matter, then number, line, plane, and body must be one and the same thing.[45] Conversely, if their matter is not one and the same, the plane cannot contain lines, and the body cannot contain either planes or lines.

The point of Aristotle's criticism is that, with respect to the different hypostatizations of the principles of "the great and the small," their introduction is inconsistent with some of the basic propositions of geometry. For example, these include the proposition that the point is on the straight line or that the straight line belongs to the plane. Aristotle's objections to Plato's theory of dimensions are the consequence of his conception of mathematical entities, which can be characterized as a mathematical substantialism or realism. His mathematical realism assumes that the dimensions belonging to the lower type of the limit (πέρας) are the dimensions of the higher type: the point is the limit of the line, the line is the limit of the plane, and the plane is the limit of the

[44] Aristotle, *Metaphysics*, 1090b32–33.
[45] Ibid., 1085a22–25, 35.

body, but they are not independent entities. The reason for this is that they do not exist separately (in abstraction); they can be ostensively referred to (τόδε τι) only within the body, but they do not satisfy the criterion to be ουσίες (substances). Conceptually, lower dimensions epistemologically precede higher ones in the sense that knowledge of the higher dimensions is based on knowledge of the lower ones. Aristotle claims: "not all things which are prior in formula are prior in substance."[46] That which is first in terms of perceptual knowledge is not the first in terms of ontology. Physical bodies are ontologically prior to geometrical objects, which are analyzed separately. Geometrical bodies are merely abstractions from physical objects. As a field of study, geometry deals with properties such as location and relation. This does not imply, however, that geometrical objects have an independent and separate reality. This simply means that geometrical objects cannot exist independently of the physical objects from which they have been abstracted.

We also need to understand Plato's viewpoint concerning the generation of dimensions from the One and the Indeterminate Dyad. A dimension is said to be καθ΄αύτο (by itself) if it partakes in an Idea. A "length by itself" would not be a "breadthless length" for it is not a unidimensional magnitude, but a concept. We cannot predicate geometrical concepts the way we do when Ideas are at stake. For example, a diagonal and the side of a square are incommensurate magnitudes. However, the Idea of the line as well as the Idea of the magnitude are themselves neither commensurate nor incommensurate, for it makes no sense to talk about Ideas in this way. Aristotle's objection refers to Plato's hypostatization of "dimensions by themselves" and his turning geometrical dimensions into separate genera. These lead to an unnecessary multiplication of entities (the logical restriction later to be named "Occam's razor"). Plato probably thought that "length by itself" would generate the reflection of the principle of the One and that of the Indeterminate Dyad (the long and the short), whereas the representation of length in geometry would be generated via the operation of the Indeterminate Dyad and the Idea of number. "The long and the short" become generative principles of the line with regard to the insight that "long" and "short" are the properties of finite lengths (line

[46] Ibid., 1077b1.

segments). The finite length is a unity which can be truthfully predicated either one of two attributes: "long" and "short."

This insight provides the answer to the dilemma posed at the very beginning of this paper concerning the possibility for the geometrical objects to be perceived (seen) in the physical world. There is a tradition of many centuries behind the proposition that geometrical objects cannot be seen. However, let us imagine a situation in which we are standing at a seashore gazing at the horizon. What we will witness is the meeting of two surfaces, namely the surface of the sea and the surface of the sky. At their meeting point, we might see the line separating the two surfaces. Since the meeting point of the surfaces has length (namely, the line segment) and has no breadth, it satisfies the conditions for being a line. It could be objected that what is seen does not count as a line, but rather merely as two differently colored surfaces! The problem is that there is a difference between seeing a color and seeing a shape. If someone says that he/she sees a green circle or a green circular surface, we immediately know that the person's field of vision also contains a surface of another color. The circular surface must be seen as finite, but the green one does not have to. "The statement about seeing a line implies the statement about seeing two surfaces of a different color or shade since the perception of a geometrical line is impossible without the perception of two surfaces that are mutually limited."[47] When we see a geometrical line, we see the way in which two surfaces limit each other. One has to bear in mind that the borderline between two colored surfaces is not the part of that which it limits. The same holds for the example of four differently colored continuously tangential squares adding up to a bigger square. If we paid attention, would we not "observe" a point at the center of this square?

10. Geometrical Proportion

The conception that the existence of geometrical objects is dependent on material extension is a thesis of a materialistic monist who fails to take into consideration the ordering of matter and shape. This hypothesis can thus be rejected. Therefore, we can accept the thesis that geometrical objects are limits and that they have a particular kind of existence. Plato's cosmogonic discourse in *Timaeus* has some overtones strongly reminis-

[47] Miloš Arsenijević, *Prostor, vreme, Zenon*, p. 17.

cent of Parmenides' (and Zeno's) One (Being), namely, among other things, that the cosmos is one, unique, and continuous.[48] Plato accepted the so-called "continuous geometrical proportion" (ἀναλογία συνεχές) through which an infinite geometrical progression is formed. For Plato, infinite geometrical progression serves as a mathematical model.[49] In geometrical progression the quotient of the two successive members is constant. Each member of the geometrical progression represents a geometrical mean of the adjoining members that is calculated according to the following formula:

$$G = \sqrt{a^1 \times a^2} \text{ for the whole series } G = \sqrt[n]{a^1 \times a^2 \times \ldots \times a^n}$$

In a geometrical proportion, in the same way in which the first member relates to the second, the second relates to the third, i.e., the three numbers *a*, *b*, and *c* form a geometrical proportion when:

$$(a : b) = (b : c)$$[50]

Plato is of the opinion that an infinite geometrical proportion generates the beings of the visible cosmos bringing them into mutual harmony. The elements are mutually distributed in a continuous geometrical proportion, and the physical body of the cosmos is thus completely and proportionally congruent.[51] Since it is continuous, the geometrical proportion serves the function of the connection (δεσμός) that provides organic unity to the cosmos. The continuous proportion enables the establishment of the links among the elements, based on which the physical body of the cosmos turns from a contingent (contiguously tangential) into a continuous whole (ὅλον).

In the last analysis, for the Ancient Greeks proportion amounts to similarities between objects. The first observation of similarities happens at the perceptual level as the observation of the same form. Quantitative similarity, i.e., proportion, is observed among line segments and this relation is secondarily expressed in numerical terms. The last source

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[48] Plato, *Timaeus*, 30d2–3.

[49] It should be reminded that Zeno's aporia "Dichotomy" can be represented via a decreasing geometrical progression.

[50] Thomas Heath, *A History of Greek Mathematics*. *Volume 1: From Thales to Euclid*, pp. 85–87.

[51] Plato, *Timaeus*, 31b6–32.

of proportion is a geometrical intuition (*Anschauung*) that completes geometry as a mediator between arithmeticals and concepts. The first elements appear in space. These four elements (earth, water, air, and fire) are primordial properties that are constantly changing.[52] Then Plato talks about the demiurge assigning geometric shapes to these elements. He begins by positing three types of triangles, from which the surfaces of solids should be built. Things become visible when they obtain a third dimension. So, first surfaces are created, and then bodies are built.[53] Plato proposes the hypothesis of the symmetry of the regular spatial figures as well as the hypothesis of the "most beautiful" elementary triangles as the basis for the construction of the universe.[54] Here, elementary triangles are the authentic atoms of the cosmic genesis. Assuming that Plato is indeed the author of *Timaeus* (which is contested), this can serve as a proof that he did not entertain the theory of the "incommensurate lines." Although *Timaeus* deals with the structure of the physical cosmos, it is evident that it is mathematically modelled. The mathematical theory of proportion plays a central role in the first hypothesis about the construction of the cosmos. The geometrical theory of proportion is essential for the second hypothesis. It turns out that the elements of Plato's cosmos were regular geometrical bodies, which simply means that the cosmos is geometrically modelled. The central concept of *Timaeus*, i.e., the concept of χώρα (*chora*), combines the classical determinations of ὕλη (*hyle*) with the geometrical determination of extension. The introduction of stereometry into cosmogony reveals that, not only did Plato ascribe geometrical structure to the cosmos, but that he also derived an endless diversity from it.[55] It was necessary that geometry should occupy a particular place in the cosmic genesis since the "soul of the cosmos" was not a material cause (consisting of the four basic elements) but rather the self-movable efficient cause of the cosmos as well as its for-

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[52] Ibid., 49e2–4.

[53] Ibid., 55d6–55 e.

[54] Ibid., 53b4. The term σύμμετρος (*summetros*) was used in Ancient Greek mathematics predominantly in the context of commensurability.

[55] Ernst Cassirer correctly notes that geometrical propositions can neither be verified nor refuted by experience. The incongruity between geometrical axioms and physical objects requires the revision of the latter. Ernst Cassirer, "Einstein's Theory of Relativity," in *Substance and Function and Einstein's Theory of Relativity*, Chicago: The Open Court Publishing Company, 1923, p. 100.

mal cause, since it introduced regularity into the cosmos.[56] The "cosmic soul" is both logically and ontologically prior to the "cosmic body." Just as geometry was constitutive for the construction of the physical cosmos, so was the Pythagorean belief that proportion and harmony constitute the basis for the order of the "cosmic soul," i.e., the form. It is clear that Plato introduced the regular spatial figures and elementary triangles (i.e., mathematical entities) into his theory of the physical structure of the cosmos, but the claim that "an endless diversity of the cosmos" is generated through geometrical shapes is a philosophical one. Mathematical entities play an important part in drawing a holistic picture of the cosmos by providing its very basis.

It is justified to ask whether the cosmogony of *Timaeus* has a scientific character, and not merely a mythological one. What counts as "scientific" is Plato's distinction between geometrical and arithmetical number via the principle of the Indeterminate Dyad ("the great and the small"). A line segment is geometrical, whereas a number is an arithmetical entity. This can be seen through the hypostatization of the generative principles of Ideas-numbers as well as through the generation of ideal magnitudes. "The great and the small" generate Ideas-numbers as well as "the much and the little" do, whereas ideal magnitudes are generated by "the long and the short," "the broad and the narrow," and "the deep and the shallow." From this, it can be concluded that the viewpoint concerning Ideas-numbers was formed under the influence of the Ancient Greek arithmetical conception of numbers.

11. Summary of the Influence of Zeno's Axiom on Plato's Geometry

The Peripatetic text *De lineis insecabilibus* contains an indication to the effect that Plato denied the real existence of points and that he rather viewed them as geometrical fictions advocating, instead, for the existence of indivisible lines (line segments). If there exists such a geometrical object called a line, then it must have a magnitude and shape. Each line is infinitely divisible. Therefore, there are no indivisible lines. There are two ways in which we can save Plato from negating Zeno's Axiom. One is to say that Plato did not advocate for the theory of indivisible lines at all, because this theory forces us to accept the claim that it is possible to measure all magnitudes, which is impossible. Another way is

[56] Georgy Vlastos, *Plato's Universe*, Las Vegas: Parmenides Publishing, 2005, p. 31.

to indicate that, in *Theaetetus*, Plato clearly understood the mathematical concept of incommensurability and that, for this reason, he could not be the author of the theory of indivisible lines. In our interpretation, Plato's doctrine of ideal dimensions refers to the Ideas of geometrical magnitudes. In the geometrical context, the term χώρα (*chora*) is understood as meaning macrocosmic order. The elementary triangles of *Timaeus* are not treated as indivisible planes, but as figures in the plane (three-sided polygons). Consequently, the elements of the physical cosmos in *Timaeus* are geometrical, but they nevertheless could not be mathematically derived. Plato's Ideas-numbers and paradigmatic dimensions are not mathematical entities but transcendental forms of cognition as well as real conditions for the possibility of abstract subjective knowledge. In a mathematical interpretation of Zeno's Axiom, Adolf Grünbaum suggested two similar formulations: (1) "the sum of an infinite number of equal positive magnitudes of arbitrary smallness must necessarily be infinite," and (2) "the sum of any finite or infinite number of 'dimensionless' magnitudes must necessarily be zero."[57]

We have accepted the formulation of Zeno's Axiom according to which nothing can be constructed beginning with a zero-dimensional entity, a nothing. On this view, points cannot constitute the multitude. Zeno thought that, in this way, the pluralistic hypothesis could be refuted, the assumption that there is a multitude being one of the main results of the negative dialectic. The acceptance of Zeno's Axiom led Aristotle to the theory of geometrical objects as continuously tangential limits. The negation of Zeno's Axiom in any of its aspects leads to topological aporias. It contradicts the (acceptable) thesis that in any limited space there is enough space for an arbitrarily big, but still limited number of points, lines, and planes. Another consequence of the acceptance of Zeno's Axiom is the infinitistic mathematical thesis. This thesis does not take into consideration the difference between a particular body with a finite number of parts and the space that this body occupies and to which (space), in addition, there belong unlimited number of parts.

There is yet one more solution to the metric and topological problems of Zeno's type arising from the paradoxes of infinity. This possibility, suggested by Arsenijević, could be called the "indefinitistic solu-

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[57] Adolf Grünbaum, "Metrical Paradox of Extension." In: Wesley C. Salmon (ed.), *Zeno's Paradoxes*, Indianapolis: Bobbs-Merrill, 1967, p. 178.

tion." According to this suggestion, there must be a finite number of actualized limits in the process of division even though the number of possible limits is infinite. In the difference between the actual and the potential, Arsenijević sees the difference between physical and mathematical space. This is enabled by the fact that the premises of the *staccato* and *legato* version of Achilles are mutually insupportable.[58] The distinction between mathematical and physical space and time leads to a more or less acceptable solution to the metric, topological, and kinematic aporias of the famous Zeno of Elea. His axiom holding that a line, being a continuum, cannot be understood as a set of points found a fruitful ground within the intuitionistic mathematics in which the geometrical parts of the lower dimensions can only be the limits of the higher-dimensional entities, i.e., the limits of the heterogeneous parts of physical bodies. There is a tendency in contemporary mathematics to implicitly define the basic geometrical objects by means of the axioms regulating their relations.[59] We understand these geometrical axioms as ideally construed *a priori* propositions and not as "images" of particularly given realities. Experience, as a synthetic *nexus* of presentation, cannot provide the foundation for geometrical axioms, since axioms must be known *a priori*. Phenomena must be evaluated according to the fundamental geometrical theses and not the other way around.

[58] Miloš Arsenijević, *Prostor, vreme, Zenon*, p. 326.

[59] David Hilbert, *Grundlagen der Geometrie*, Dresden: Fachbuchverlag, 2015, Chapter 1. A similar procedure using "implicit definitions" can also be found in the fifth Postulate of Euclid's *Elements* in which the straight line is defined. That definition is mathematically possible only if the system (set) of axioms of this geometry is self-consistent.